# Backstepping boundary stabilization of a cross-diffusion system in a one-dimensional moving domain: linearized system

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# Outline of the talk

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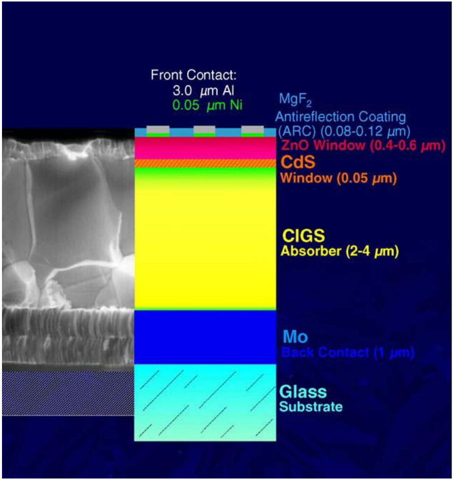
# Motivation: thin film CIGS solar cell production

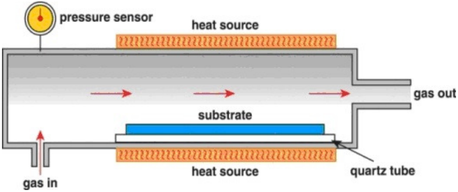
Collaboration with IRDEP (Institut de Recherche et Développement sur l'Energie Photovoltaïque, EDF, CNRS, Chimie Paristech).

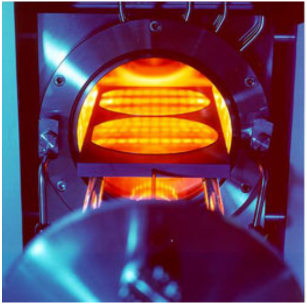
Optimal control of the production process of thin film CIGS (Copper, Indium, Gallium, Selenium) solar cell devices

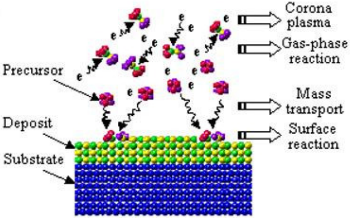
# Typical composition of a CIGS solar cell



# Production process: Physical Vapor Decomposition (CVD)



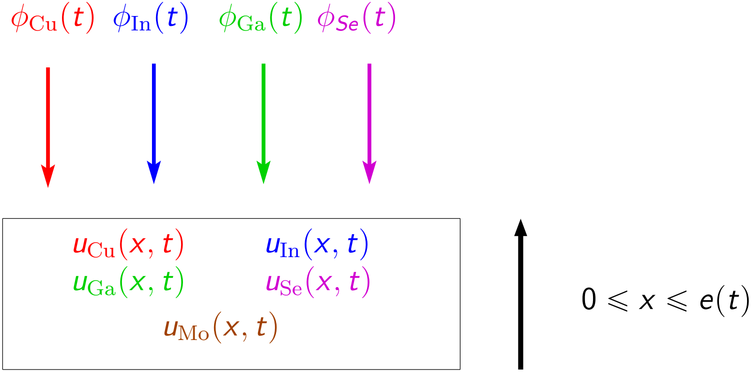


# Simplified 1d model

Let us denote by $\mathcal{A} := \{\mathrm{Cu}, \mathrm{In}, \mathrm{Ga}, \mathrm{Se}, \mathrm{Mo}\}$ the set of the different atomic species involved in the process and by $e(t)$ the thickness of the solid at time $t$.

The production process lasts during a time $T > 0$.

The aim is to control the fluxes of the different atomic species $(\phi_A(t))_{A\in\mathcal{A}, 0\leqslant t\leqslant T}$ in order to obtain a desired thickness of the cell $e^{\mathrm{opt}}(T)$ and desired profile of concentrations $(u_A^{opt}(x))_{A\in\mathcal{A},\ 0\leqslant x\leqslant e^{\mathrm{opt}}(T)}$ at time $t = T$.



# Modelisation

One has to take into account:

- the cross-diffusion phenomena occuring inside the bulk between the different chemical species;
- the evolution of the surface of the solid due to the imposition of the external fluxes.

# A one-dimensional cross-diffusion model [Bakhta,VE,2018]

- $n+1$ chemical species with volumic fractions $u_0(t,x),\dots,u_n(t,x)$, and constraints:

$$\forall\ 0 \leqslant i \leqslant n,\ \ u_i(t,x) \geqslant 0 \text{ and } \sum_{j=0}^{n} u_j(t,x) = 1.$$

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- 1d domain $(0, e(t)) \subset \mathbb{R}_+$, $e(t) > 0$ is the thickness of the film. The evolution of the domain is determined by the incoming external fluxes $(\phi_0(t), \dots, \phi_n(t))$ absorbed at the extremity $x = e(t)$:

$$e(t) := e_0 + \int_0^t \sum_{i=0}^{n} \phi_i(s) ds.$$

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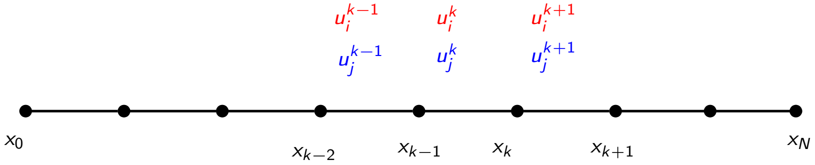
- Cross-diffusion equations formally derived from a stochastic hopping model on a network: for $0 \leqslant i \leqslant n$,

$$\partial_t u_i - \partial_x \left( \sum_{j=0}^{n} K_{ij} \left( u_j \partial_x u_i - u_i \partial_x u_j \right) \right) = 0,\ \ x \in (0, e(t)).$$

for some coefficients $K_{ij} = K_{ji} > 0$.

# Formal (1d) derivation of the model for the cross-diffusion equations inside the solid




Let $\Delta t > 0$. Let $p_{ij} = p_{ji}$ represent the probability that an atom of type $i$ exchange its position in the solid with an atom of type $j$.

Assume for the moment that $d = 1$, $\Omega = (0, e_0)$ with $e_0 > 0$ and let us introduce a uniform discretization grid $(x_k)_{0 \leqslant k \leqslant N}$ of $(0, e_0)$ defined by $x_k = k\Delta x$ with $\Delta x = \frac{e_0}{N}$ for $N \in \mathbb{N}^*$.

Let us denote by $u_i^k(t)$ the local concentration of atom $i$ in the $k^{th}$ cell $(x_{k-1}, x_k)$.

$$
\begin{aligned}
& u_i^k(t+\Delta t) - u_i^k(t) \\
\approx & \sum_{0 \leqslant j \leqslant n,\ j \neq i} p_{ij}[u_j^k(t)u_i^{k+1}(t) + u_j^k(t)u_i^{k-1}(t) - u_j^{k-1}(t)u_i^k(t) + u_j^{k+1}(t)u_i^k(t)] \\
\approx & \sum_{0 \leqslant j \leqslant n,\ j \neq i} p_{ij}[u_j^k(t)(u_i^{k+1}(t) + u_i^{k-1}(t) - 2u_i^k(t)) - u_i^k(t)(u_j^{k-1} + u_j^{k+1}(t) - 2u_j^k(t))]
\end{aligned}
$$

# Formal derivation of the model for the diffusion equations inside the solid

Taking now (for instance) the scaling $\Delta t = 2Q\Delta x^2$ for some constant $Q > 0$ and denoting by $K_{ij} := \frac{p_{ij}}{Q}$, we obtain the limit equation

$$\begin{aligned}\partial_t u_i &= \sum_{0\leqslant j\leqslant n,\ j\neq i} K_{ij}(u_j \partial_{xx} u_i - u_i \partial_{xx} u_j) \\ &= \partial_x \left[ \sum_{0\leqslant j\leqslant n,\ j\neq i} K_{ij} \left( u_j \partial_x u_i - u_i \partial_x u_j \right) \right].\end{aligned}$$

**Remark:** Rigorous hydrodynamic limit of multi-species symmetric exclusion systems [Quastel, 91], [Erignoux, 2018], [Dabaghi, VE, Strössner, 2018]

# A one-dimensional cross-diffusion model (2)

- Writing $u_0 = 1 - \sum_{i=1}^n u_i$, the unknown vector $u := (u_1, \ldots, u_n)^T$ solves the system:

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with the (non-symmetric) diffusion matrix

$$A_{ii}(u) = \sum_{1 \leqslant j \neq i \leqslant n} (K_{ij} - K_{i0})u_j + K_{i0}, \text{ and } A_{ij}(u) = -(K_{ij} - K_{i0})u_i.$$

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- The absorption condition at the extremity $x = e(t)$ reads: $(\varphi := (\phi_1, \ldots, \phi_n)^T)$

$$(A(u)\partial_x u)(t, e(t)) + e'(t)u(t, e(t)) = \varphi(t).$$

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- The complete system reads

$$\begin{cases} \partial_t u - \partial_x(A(u)\partial_x u) = 0, & \text{for } t \in (0, +\infty), \; x \in (0, e(t)), \\ (A(u)\partial_x u)(t, 0) = 0, & \text{for } t \in (0, +\infty), \\ (A(u)\partial_x u)(t, e(t)) + e'(t)u(t, e(t)) = \varphi(t), & \text{for } t \in (0, +\infty), \\ u(0, x) = u^0(x), & x \in (0, e_0), \end{cases}$$

# Formal gradient flow structure: case of zero flux boundary conditions

[Jordan, Kinderlehrer, Otto, 1998], [Burger, Di Francesco, Pietschmann, Schlake (2010)], [Zinsl, Matthes, 2014], [Mielke, Liero, 2013], [Jungel, 2015], [Daneri, Savaré, 2010]

**Special case when $\varphi = 0$ and $e(t) = e_0$ for all $t \geqslant 0$**

Let $S := \{u = (u_i)_{1 \leqslant i \leqslant n} \in (\mathbb{R}_+^*)^n \mid \sum_{1 \leqslant i \leqslant n} u_i < 1\}$.

Then, the system can be formally rewritten equivalently under the form

$$\partial_t u = \partial_x \left[ M(u) \partial_x Dh(u) \right], \quad u(t,x) = (u_i(t,x))_{1 \leqslant i \leqslant n} \tag{1}$$

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- $u : \left\{ \begin{array}{ccc} (0,T)\times(0,e_0) & \to & \overline{S} \\ (t,x) & \mapsto & (u_i(t,x))_{1\leqslant i\leqslant n} \end{array} \right.$
- $$h(u) := \int_{(0,e_0)} \sum_{1\leqslant i\leqslant n} u_i \ln u_i + \rho_u \ln \rho_u$$

  is the driving **entropy functional** of the system; $\rho_u = 1 - \sum_{i=1}^n u_i$

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  is the driving **entropy functional** of the system; $\rho_u = 1 - \sum_{i=1}^n u_i$
- $M(u) := \left(M_{ij}(u)\right)_{1\leqslant i,j\leqslant n}$ is the **mobility matrix** (symmetric non-negative semi-definite) defined by
  $$\forall 1 \leqslant i \leqslant n, \quad M_{ii}(u) := K_{i0} u_i \rho_u + \sum_{1\leqslant j\leqslant n,\ j\neq i} K_{ij} u_i u_j \quad \text{and}$$
  $$\forall 1 \leqslant j \leqslant n,\ j \neq i, \quad M_{ij}(u) := -K_{ij} u_i u_j.$$

The essential remark is that the entropy functional is a **Lyapunov function** for the system:

$$\begin{aligned}\frac{d}{dt}h(u) &= \int_{(0,e_0)} \partial_t u \; Dh(u)\\ &= \int_{(0,e_0)} \partial_x \left(M(u)\partial_x Dh(u)\right) Dh(u)\\ &= -\int_{(0,e_0)} (\partial_x Dh(u))^T M(u)\partial_x Dh(u)\\ &\leqslant 0.\end{aligned}$$

# Analysis of the nonlinear system: boundedness by entropy principle

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&\leqslant 0.
\end{aligned}$$

Global existence results obtained using the so-called boundedness by entropy principle, first introduced in [Burger, Di Francesco, Pietschmann, Schlake (2010)], and then further developped in [Jüngel (2015)].

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- Existence of weak solutions $u$ in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1((0, e(t)); \mathbb{R}^n))$ with $\partial_t u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; (H^1((0, e(t)); \mathbb{R}^n))')$, for fluxes $\phi_i \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. Uniqueness open in general

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- Long-time asymptotics in the case of **constant fluxes** (entropy methods):
  If the fluxes are constant in time $\phi_i(t) = \overline{\phi_i}$, defining the constant states $\overline{u_i} = \frac{\overline{\phi}_i}{\sum_{j=0}^n \overline{\phi}_j}$, the speed $\overline{v} := \sum_{i=0}^n \overline{\phi}_i$ and $\overline{e}(t) = e_0 + \overline{v}t$ , then $u_i$ converges to $\overline{u}_i$ in an average $L^1$ sense:
  $$\frac{1}{\overline{e}(t)} \|u_i(t,\cdot) - \overline{u}_i\|_{L^1(0,\overline{e}(t))} \leqslant \frac{C}{\sqrt{t+1}}.$$
- Constant "stationary states": Of course, if $u^0 = \overline{u}$ with $\overline{u} = (\overline{u}_i)_{1 \leqslant i \leqslant n}$ and $\phi_i(t) = \overline{\phi}_i$, then $u(t) = \overline{u}$ for all $t > 0$ is a solution.
  In this sense, the solution $(\overline{u}, \overline{e}(t))$ can be seen as a **constant "stationary" state** of the moving boundary problem with constant fluxes $\overline{\phi}_i$.

# Outline of the talk

# Boundary stabilization problem

$$
\begin{cases}
\partial_t u - \partial_x (A(u)\partial_x u) = 0, & \text{for } t \in (0,+\infty),\ x \in (0, e(t)), \\
(A(u)\partial_x u)(t,0) = 0, & \text{for } t \in (0,+\infty), \\
(A(u)\partial_x u)(t,e(t)) + e'(t)u(t,e(t)) = \varphi(t) := (\phi_1, \dots, \phi_n)^T, & \text{for } t \in (0,+\infty), \\
u(0,x) = u^0(x), & x \in (0, e_0),
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- Experimentalists are interested in controlling the incoming fluxes $(\phi_i)_{0 \leqslant i \leqslant n}$ to reach a target composition in the film. $\rightarrow$ difficult controllability problem.

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- **First objective**: exponential stabilization of constant "stationary states": find feedback laws $\phi_i(t) = \Phi_i(t,u(t),e(t))$ such that the system is well-posed and if $\|u^0 - \overline{u}\|_{L^2((0,e_0))}$ and $|e_0 - \overline{e}_0|$ are small enough then for some $\lambda > 0$

$$\|u(t,\cdot) - \overline{u}\|_{L^2((0,\overline{e}(t))} \leqslant Ce^{-\lambda t}\|u^0 - \overline{u}\|_{L^2((0,\overline{e}_0))},$$
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- **Strategy**: show the result for the linearized system (w.r.t $u$ and $e(t)$) around the target state $(\overline{u},\overline{e})$. (The result will follow locally for the nonlinear system if we can control the nonlinear terms.)

# Linearized system around $(\overline{u}, \overline{e})$ and choice of control variables

- Small perturbations $(\delta u^0, \delta e_0)$ at $t = 0$ around the initial condition $\overline{u}$ and initial thickness $\overline{e}_0$:

$$u^0 = \overline{u} + \delta u^0$$
$$e_0 = \overline{e}_0 + \delta e_0$$

- Small perturbations $(\delta \phi^i(t))$ of the fluxes at times $t$ around $\overline{\phi}_i$:

$$\phi_i(t) = \overline{\phi}_i + \delta \phi_i(t)$$

- Induce perturbations $(\delta u(t), \delta e(t))$ on $u(t)$ and $e(t)$ at time $t$:

$$u(t) = \overline{u} + \delta u(t),$$
$$e(t) = \overline{e}(t) + \delta e(t)$$

- Investigate the linearized dynamic of $(\delta u(t), \delta e(t))$, the first-order corrections of $(u(t) - \overline{u}, e(t) - \overline{e}(t))$.

# Linearized system around $(\overline{u}, \overline{e})$ and choice of control variables

- **Linearized system:**

$$\delta e(t) = \int_0^t \sum_{i=0}^{n} \delta\phi_i(s)\,ds + \delta e_0, \;\; t \geqslant 0,$$

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and:

$$\begin{cases} \partial_t \delta u - A(\overline{u})\partial^2_{xx}\delta u = 0, & \text{for } t \in (0,+\infty), \ x \in (0, \overline{e}(t)), \\ A(\overline{u})\partial_x \delta u(t, \overline{e}(t)) + \overline{v}\delta u(t, \overline{e}(t)) = \delta\eta(t), & \text{for } t \in (0,+\infty), \\ A(\overline{u})\partial_x \delta u(t, 0) = 0, & \text{for } t \in (0,+\infty), \\ \delta u(0,x) = \delta u^0(x), & \text{for } x \in (0, \overline{e}_0). \end{cases}$$

where

$$\delta\eta(t) := \delta\varphi(t) - \delta e'(t)\overline{u} \in \mathbb{R}^n.$$

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- New control variables: $(\delta\phi_0, \dots, \delta\phi_n) \to (\delta\theta := \sum_{i=0}^n \delta\phi_i, \delta\eta_1, \dots, \delta\eta_n)$. $\to$ Control analysis is decoupled: $\delta\theta$ controls the thickness $\delta e$ and $\delta\eta$ controls the volumic fractions $\delta u$.

# Motivation for backstepping

$$\begin{cases} \partial_t \delta u - A(\overline{u})\partial^2_{xx}\delta u = 0, & \text{for } t \in (0,+\infty),\ x \in (0,\overline{e}(t)), \\ A(\overline{u})\partial_x \delta u(t,\overline{e}(t)) + \overline{v}\delta u(t,\overline{e}(t)) = \delta\eta(t), & \text{for } t \in (0,+\infty), \\ A(\overline{u})\partial_x \delta u(t,0) = 0, & \text{for } t \in (0,+\infty), \end{cases}$$

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- Assuming the non degeneracy of the target state (i.e. $\overline{u}_i > 0$ for any $1 \leqslant i \leqslant n$ and $1 - \sum_{j=1}^n \overline{u}_j > 0$), we show using the entropic structure of the system that $A(\overline{u})$ can be diagonalized with positive eigenvalues (altough non-symmetric). ($\rightarrow$ Can be generalized to any entropic systems with symmetric mobility matrix).

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- At the end of the day we are left to stabilize $n$ independent scalar problems of the form:

$$\begin{cases} \partial_t z - \sigma\partial^2_{xx} z = 0, & \text{for } t \in (0,+\infty),\ x \in (0,\overline{e}(t)), \\ \sigma\partial_x z(t,\overline{e}(t)) + \overline{v} z(t,\overline{e}(t)) = \delta\psi(t), & \text{for } t \in (0,+\infty), \\ \sigma\partial_x z(t,0) = 0, & \text{for } t \in (0,+\infty). \end{cases}$$

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- At the end of the day we are left to stabilize $n$ independent scalar problems of the form:

$$\begin{cases} \partial_t z - \sigma\partial_{xx}^2 z = 0, & \text{for } t \in (0,+\infty),\ x \in (0,\overline{e}(t)), \\ \sigma\partial_x z(t,\overline{e}(t)) + \overline{v}z(t,\overline{e}(t)) = \delta\psi(t), & \text{for } t \in (0,+\infty), \\ \sigma\partial_x z(t,0) = 0, & \text{for } t \in (0,+\infty). \end{cases}$$

- Basic $L^2$ Lyapunov functions of the form

$$V(t) := \int_0^{\overline{e}(t)} f(x) z(t,x)^2(t,x)\,dx$$

fail to show exponential stability.

# Outline of the talk

# Backstepping: main idea

- Backstepping technique has been developed since the 2000's (see [Krstic,Smyshlyaev,2008] for instance), and recently used for the heat equation with variable coefficients [Coron, Nguyen, 2017].

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# Backstepping: main idea

- Backstepping technique has been developed since the 2000's (see [Krstic,Smyshlyaev,2008] for instance), and recently used for the heat equation with variable coefficients [Coron, Nguyen, 2017].
- **Idea**: transform the system into a target system that is shown to be stable. Use the reverse transformation to show stability of the original system.
- We want to transform our system to the same one with homogeneous boundary conditions and an additional damping term with $\lambda > 0$ (provide $\lambda$-exponential stability in $L^2$):

$$
\begin{cases}
\partial_t g_\lambda - \sigma \partial^2_{xx} g_\lambda + \lambda g_\lambda = 0, & \text{for } t \geqslant 0,\ x \in (0, \overline{e}(t)), \\
\sigma \partial_x g_\lambda(t, \overline{e}(t)) + \overline{v} g_\lambda(t, \overline{e}(t)) = 0, & \text{for } t \in (0, +\infty), \\
\sigma \partial_x g_\lambda(t, 0) = 0, & \text{for } t \in (0, +\infty), \\
g_\lambda(0, \cdot) = g^0_\lambda = z^0_\lambda. &
\end{cases}
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- There will be an associated feedback flux $\delta\psi_\lambda$ such that

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# Backstepping: Volterra transform and stability

- Consider a Volterra transform of the second kind, of the form:

$$g_\lambda(t,x) = (\mathcal{T}_\lambda z_\lambda)(t,x) := z_\lambda(t,x) - \int_0^x k_\lambda(t,x,y) z_\lambda(t,y)dy, \ \ t \geqslant 0, \ \ x \in (0, \overline{e}(t)),$$

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that can be shown to be invertible if $k_\lambda(t,\cdot,\cdot) \in L^2(D_t)$, where

$$D_t := \left\{ (x,y) \in (\mathbb{R}_+)^2, \quad 0 < y \leqslant x < \overline{e}(t) \right\},$$

with inverse of the same form:

$$z_\lambda(t,x) = (\mathcal{T}_\lambda^{\mathrm{inv}} g_\lambda)(t,x) := g_\lambda(t,x) + \int_0^x l_\lambda(t,x,y) g_\lambda(t,y)\,dy, \;\; t \geqslant 0, \;\; x \in (0, e(t)),$$

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- As a consequence, if $g_\lambda$ is a solution of the (stable) target system, the following stability loop holds:

$$\begin{aligned} ||z_\lambda(t)||_{L^2((0,e(t))} &\leqslant \left(1 + ||l_\lambda(t)||_{L^2(D_t)}\right) ||g_\lambda(t)||_{L^2(0,e(t))} \\ &\leqslant C \left(1 + ||l_\lambda(t)||_{L^2(D_t)}\right) e^{-\lambda t} ||g_\lambda(0)||_{L^2((0,e_0))} \\ &= C \left(1 + ||l_\lambda(t)||_{L^2(D_t)}\right) e^{-\lambda t} ||z_\lambda(0)||_{L^2((0,e_0))} \end{aligned}$$

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- **Question: What must satisfy the kernels $k_\lambda, l_\lambda$ and the feedback $\delta\psi_\lambda$ for the backstepping transformation $\mathcal{T}_\lambda$ to map the original system to the target system, and vice-versa ?**

# Backstepping: feedback and kernel equations

- The form of the feedback is dictated by the mapping of the boundary conditions:

$$\delta\psi_\lambda(t) := \sigma k_\lambda(t,\overline{e}(t),\overline{e}(t)) z_\lambda(t,\overline{e}(t)) + \int_0^{\overline{e}(t)} \left[\sigma \partial_x k_\lambda(t,\overline{e}(t),y) + \overline{v} k_\lambda(t,\overline{e}(t),y)\right] z_\lambda(t,y)\,dy.$$

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- For $\mathcal{T}_\lambda$ to map the original PDE to the target PDE, $k_\lambda$ must satisfy the kernel equations:

$$\begin{cases} \partial_{xx}^2 k_\lambda(x,y) - \partial_{yy}^2 k_\lambda(x,y) = \dfrac{\lambda}{\sigma} k_\lambda(x,y) & (x,y)\in\{0<y\leqslant x<+\infty\}, \\ \partial_y k_\lambda(x,0) = 0 & x\in(0,+\infty), \\ \dfrac{d}{dx}k_\lambda(x,x) = -\dfrac{\lambda}{2\sigma} & x\in(0,+\infty), \\ k_\lambda(0,0) = 0, & \end{cases}$$

**Good news:** $k_\lambda$ does not depend on the time variable here!

# Backstepping results: scalar case

[Cauvin-Vila, VE, Hayat, 2022]

Finally, we show that:

- The kernel equations are well-posed and we have appropriate estimates on the $L^2$ norm in time: growth at most in $C_\lambda e^{ct}$ with $c$ independent on $\lambda$.
- The kernel is regular enough to define the feedback, and the linearized system is well-posed with this feedback.

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**Theorem (Rapid stabilization in $L^2$)**

*There exists $c > 0$ such that for any $\lambda > 0$ and for any non-degenerate target state $(\overline{u}, \overline{e})$, there exists a well-defined explicit feedback $\delta\psi_\lambda(t)$ (given by the previous formula) such that*

$$\|z_\lambda(t)\|_{L^2(0,\overline{e}(t))} \leqslant C_\lambda e^{(c-\lambda)t}$$

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- and furthermore: the convergence can happen in any finite time $T > 0$.

[Cauvin-Vila, VE, Hayat, 2022]

Remember that, at first order,

$$\delta u(t) \approx u(t) - \overline{u}$$
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**Corollary (Rapid stabilization in $L^2$ for the linearized system)**

*For any $\lambda > 0$, there exist well-defined explicit feedback fluxes $\left(\delta\phi_{i,\lambda}(t)\right)_{0\leqslant i\leqslant n}$ such that the linearized system is well-posed and there exists $C = C(\overline{u},\lambda)$ such that*

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# Backstepping results: linearized system case

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The convergence can also happen in any finite time $T > 0$ using the previous results in the scalar case.

# Outline of the talk

# Summary and perspectives

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- **Limit**: favorable structure of the system for which the linearized system can be uncoupled. In general, we do not know how to address the matrix kernel equation.
- **Natural development**: local stabilization of the nonlinear system.
- **More developments**: numerical tests, controllability, optimal control...

**Thank you for your attention!**